Lecture 9
Robinson instability. Vlasov equation

January 24, 2019

Lecture outline

- Robinson instability
- Vlasov equation

The Robinson instability can occur when a bunch in the ring interacts with the impedance of the fundamental mode or HOM of accelerating cavities. We will derive the condition for the instability using simple heuristic arguments, without solving the whole beam dynamics problem. A rigorous solution can be found in A. Chao's book.

Longitudinal dynamics in circular accelerator (no wakes)

The revolution frequency in a ring, $\omega_r = 2\pi/T$, depends on the particle energy,

$$\omega_r = \omega_0(1 - \alpha\eta) \tag{9.1}$$

where α is the (linear) *slip factor*[26]. Without wakes, the energy change is due to the RF force. Assuming a parabolic profile of RF potential well (the bunch length is much shorter than the RF wavelength) the RF restoring force is linear in z. The longitudinal motion is governed by the following equations:

$$\frac{dz}{dt} = -c\alpha\eta, \qquad \frac{d\eta}{dt} = \frac{1}{\alpha}\frac{\omega_{s0}^2}{c} z \tag{9.2}$$

Here $\eta = \Delta P/P \approx \Delta\mathcal{E}/\mathcal{E}$, z is the longitudinal coordinate of a particle in the bunch, ω_{s0} is the synchrotron frequency, and α is the slip factor

$$\alpha = \alpha_0 - \gamma^{-2} \tag{9.3}$$

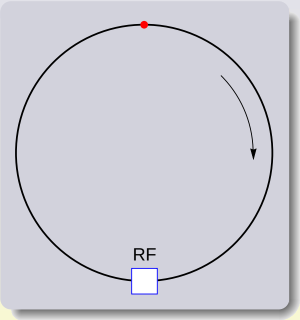
with α_0 the momentum compaction factor. In linacs $\alpha_0 = 0$.
Combining the two longitudinal equations we obtain the equation for the synchrotron oscillations

$$\ddot{z} + \omega_{s0}^2 z = 0, \qquad \ddot{\eta} + \omega_{s0}^2 \eta = 0 \tag{9.4}$$

[26] The standard notation for the slip factor is η, but we use η to denote the relative energy deviation of a particle.

Robinson instability

Consider a single bunch traveling in a ring and a cavity that has impedance $Z_\ell(\omega)$. The cavity sees a sequence of pulses separated by the revolution period. Consider M revolutions and find the energy change of the beam due to the cavity's longitudinal wake.



The "beam" charge distribution at the location of the cavity is represented by

$$\lambda(z) = \sum_{k=-M/2}^{M/2} \delta(z - kC) \qquad (9.5)$$

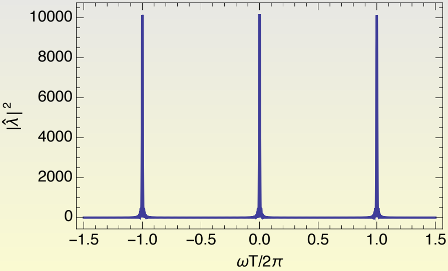
This is actually the same bunch passing M times through the cavity (note that the bunch is considered as a point charge).

Robinson instability

We now use Eq. (4.12) to find the energy change of the beam.

$$\hat{\lambda}(\omega) = \int_{-\infty}^{\infty} dz\, \lambda(z) e^{i\omega z/c} = \sum_{k=-M/2}^{M/2} e^{ik\omega T}$$

where $T = C/c = 2\pi/\omega_r$ is the revolution period. What is the function $|\hat{\lambda}(\omega)|^2$? This is a periodic function with the period ω_r. Here is the plot for $M = 100$.



The area under each peak (when integrated over ω) is $2\pi M/T$ (verified by numerical integration). It can be proved mathematically[27] that for $M \gg 1$

$$|\hat{\lambda}(\omega)|^2 \approx \frac{2\pi}{T} M \sum_{p=-\infty}^{\infty} \delta(\omega - p\omega_r) \tag{9.6}$$

[27] Compare with Eq. (8.7).

Robinson instability

Hence the energy change of the bunch is

$$\Delta\mathcal{E}_{b,M} = -N\frac{Ne^2}{\pi}\int_0^\infty d\omega \operatorname{Re} Z_\ell(\omega)|\hat{\lambda}(\omega)|^2 = -\frac{2}{T}MQ^2\sum_{p=1}^{\infty}\operatorname{Re} Z_\ell(p\omega_r)$$

where $Q = Ne$ is the bunch charge (note that $\operatorname{Re} Z_\ell(0) = 0$). Since the beam passes M times during time interval MT the energy change per unit time is

$$\Delta\dot{\mathcal{E}}_b = \frac{1}{MT}\Delta\mathcal{E}_{b,M} = -2\frac{Q^2}{T^2}\sum_{p=1}^{\infty}\operatorname{Re} Z_\ell(p\omega_r) \tag{9.7}$$

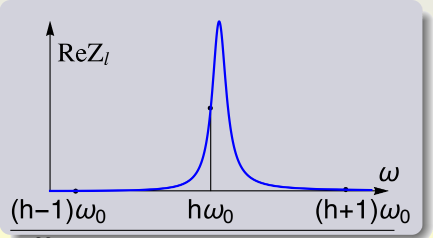
[This immediately follows from Eq. (4.19) because $\tilde{\lambda}(\omega) = 1$.]
This energy loss of the bunch is compensated by the energy gain in the cavity, QV_{acc}. The cavity is tuned to compensate for (9.7) with $\omega_r = \omega_0$—the nominal revolution frequency (corresponding to the nominal energy). That is $\omega_{RF} = h\omega_0$, h is the harmonic number.

Robinson instability

Now take into account that a variation of energy changes the revolution frequency, Eq. (9.1). The uncompensated part is[28]

$$\Delta\dot{\mathcal{E}}_b\Big|_{uncomp} = -2\frac{Q^2}{T^2}\sum_{p=0}^{\infty}\mathrm{Re}\,Z_\ell(p\omega_0(1-\alpha\eta)) + 2\frac{Q^2}{T^2}\sum_{p=0}^{\infty}\mathrm{Re}\,Z_\ell(p\omega_0)$$
$$\approx 2\frac{Q^2}{T^2}\alpha\eta\omega_0\sum_{p=0}^{\infty}p\frac{d\mathrm{Re}\,Z_\ell}{d\omega}\Big|_{p\omega_0} \qquad (9.8)$$

Here we have assumed that the deviation η is small and used the Taylor expansion.



Let us consider a situation when the revolution frequency is much larger than the width of the impedance peak, so that only one value of p plays a role. For the fundamental mode in the cavity this is the harmonic number $p = h$.

[28] Here we ignore the changes in $T = 2\pi/\omega_r$; this turns out to be small.

Robinson instability

We have $\Delta(\dot{\mathcal{E}}_b/N)/\mathcal{E}_0 = d\eta/dt$ and find

$$\frac{d\eta}{dt} = 2\eta \frac{Q^2}{NT^2\gamma mc^2} \alpha h\omega_0 \frac{d\mathrm{Re}\, Z_\ell}{d\omega}\bigg|_{\omega=h\omega_0} \equiv \frac{2}{\tau}\eta \tag{9.9}$$

where

$$\frac{1}{\tau} = \alpha \frac{Q^2 h\omega_0}{NT^2\gamma mc^2} \frac{d\mathrm{Re}\, Z_\ell}{d\omega}\bigg|_{\omega=h\omega_0} \tag{9.10}$$

We now need to go back to the equations of motion (9.2) and modify the second equation

$$\frac{d\eta}{dt} = \frac{1}{\alpha}\frac{\omega_{s0}^2}{c} z + \frac{2}{\tau}\eta$$

Combining this with the equation for z we find

$$\ddot{\eta} - \frac{2}{\tau}\dot{\eta} + \omega_{s0}^2 \eta = 0$$

Seek solution $\eta \propto e^{-i\omega t}$.

Robinson instability

We find

$$-\omega^2 + \frac{2i}{\tau}\omega + \omega_{s0}^2 = 0$$

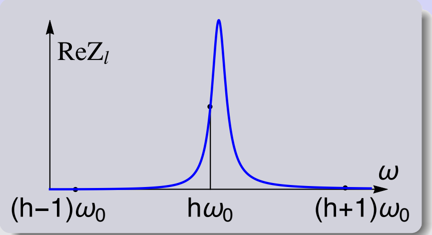
Assuming that $\omega \approx \omega_{s0} + \Delta\omega$ with $|\Delta\omega| \ll \omega_{s0}$ we find

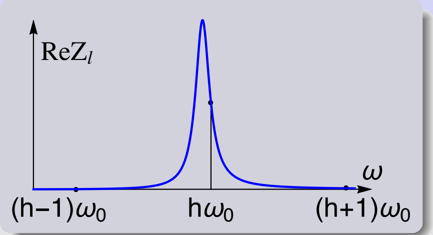
$$\Delta\omega = \frac{i}{\tau} \tag{9.11}$$

A positive imaginary part of ω (that is $\tau > 0$) means an instability with the growth time τ. This occurs if $d\mathrm{Re}\, Z_\ell/d\omega|_{\omega=h\omega_0} > 0$. This is the *Robinson instability*.

In our simplified derivation we assumed that the particle arrives at the location of the cavity with the period $2\pi/\omega_r$, ignoring the fact that the beam oscillates longitudinally with the frequency ω_{s0}. Hence our result is actually valid if ω_{s0} is much smaller than the width of the resonant impedance. If this does not hold, the impedance in Eq. (9.7) will be sampled at the frequencies $p\omega_r \pm \omega_{s0}$.

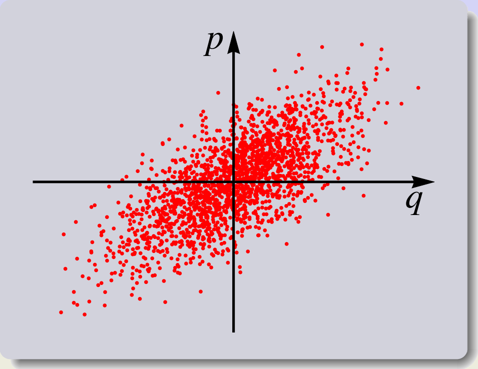
Robinson instability





Left plot—unstable beam (above the transition energy); right plot—stable beam.

Vlasov equation

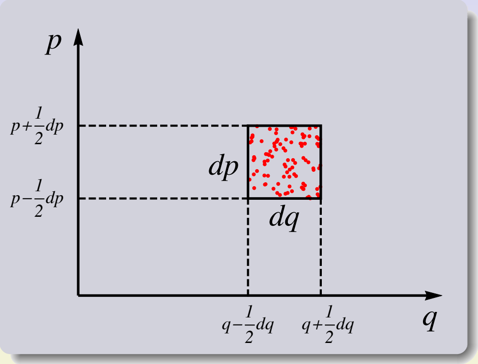


Instead of thinking about the beam as a collection of discrete particles numbered from 1 to N, $i = 1, 2, \ldots, N$, in Vlasov formalism, one represents a beam as a "fluid" in a 6-dimensional phase space (for 3 degrees of freedom). The beam dynamics is described by the time evolution of the "fluid" density. This density function satisfies the Vlasov equation.

The Vlasov, or *kinetic*, equation is extremely powerful technique, that can be used for study of beam stability, intra-beam scattering, quantum diffusion effects, etc. At the same time it is more complicated than a typical single-particle analysis often used in accelerator physics for simpler problems.

Distribution function in phase space and kinetic equation

We start from considering a simple case of one degree of freedom with the canonically conjugate variables q and p (it may be $x - x'$, $y - y'$ of $z - \eta$ pair).



Consider an infinitesimally small region in phase space $dq \times dp$ and let the number of particles of the beam at time t in this phase space element be given by dN. Mathematically infinitesimal phase element should be physically large enough to include many particles, $dN \gg 1$. We define the distribution function of the beam $f(q, p, t)$ such that

$$dN(t) = f(q, p, t)\,dp\,dq\,. \tag{9.12}$$

We can say that the distribution function gives the *density* of particles in the phase space.

Distribution function in phase space and kinetic equation

Particles travel from one place in the phase space to another, and the distribution function evolves with time. Our goal is to derive the *kinetic equation* that governs this evolution. In this derivation, we will assume that the particle motion is Hamiltonian. We assume that we are given the rate of change of the coordinate and momentum as functions of q, p and t: $\dot{q}(q, p, t)$ and $\dot{p}(q, p, t)$.

The number of particles in this region at time t is given by Eq. (9.12). At time $t + dt$ this number will change because of the flow of particles through the boundaries. Due to the flow in the q-direction the number of particles that flow in through the left boundary is

$$f\left(q - \frac{1}{2}dq, p, t\right) \times dp\,\dot{q}\left(q - \frac{1}{2}dq, p, t\right) \times dt \tag{9.13}$$

and the number of particles that flow out through the right boundary is

$$f\left(q + \frac{1}{2}dq, p, t\right) \times dp\,\dot{q}\left(q + \frac{1}{2}dq, p, t\right) \times dt\,. \tag{9.14}$$

Distribution function in phase space and kinetic equation

Similarly, the number of particles which flow in through the lower horizontal boundary is

$$f\left(q, p - \frac{1}{2}dp, t\right) \times dq\, \dot{p}\left(q, p - \frac{1}{2}dp, t\right) \times dt \tag{9.15}$$

and the number of particles that flow out through the upper horizontal boundary is

$$f\left(q, p + \frac{1}{2}dp, t\right) \times dq\, \dot{p}\left(q, p + \frac{1}{2}dp, t\right) \times dt\,. \tag{9.16}$$

The number of particles in the volume $dq \times dp$ is now changed

$$\begin{aligned} & dN(t + dt) - dN(t) \\ & = [f(q, p, t + dt) - f(q, p, t)]dp\, dq \\ & = \end{aligned} \tag{9.17}$$

Distribution function in phase space and kinetic equation

$$
\begin{aligned}
&= f\left(q - \frac{1}{2}dq, p, t\right) dp\, \dot{q}\left(q - \frac{1}{2}dq, p, t\right) dt \\
&- f\left(q + \frac{1}{2}dq, p, t\right) dp\, \dot{q}\left(q + \frac{1}{2}dq, p, t\right) dt \\
&+ f\left(q, p - \frac{1}{2}dp, t\right) dq\, \dot{p}\left(q, p - \frac{1}{2}dp, t\right) dt \\
&- f\left(q, p + \frac{1}{2}dp, t\right) dq\, \dot{p}\left(q, p + \frac{1}{2}dp, t\right) dt
\end{aligned}
\tag{9.18}
$$

Dividing this equation by $dp\, dq\, dt$ and expanding in Taylor's series (keeping only linear terms in dp, dq, dt) gives the following equation

$$
\frac{\partial f}{\partial t} + \frac{\partial}{\partial q}\dot{q}(q, p, t) f + \frac{\partial}{\partial p}\dot{p}(q, p, t) f = 0 \,. \tag{9.19}
$$

What we derived is the *continuity* equation for the function f.

Incompressible Hamiltonian flow

Due to the Hamiltonian nature of the flow in the phase space a medium represented by a distribution function f is *incompressible*. This follows from the Liouville theorem. According to this theorem the volume of a space phase element does not change in Hamiltonian motion. Since the value of f is the number of particles in this volume, and this number is conserved, f within a *moving* elementary volume is also conserved. The density at a given point of the phase space q, p however changes because other liquid elements arrive at this point at a later time.

Mathematically, the fact of incompressibility is reflected in the following transformation of the continuity equation (9.19). Let us take into account the Hamiltonian equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$:

$$\frac{\partial}{\partial q}\dot{q}(q,p,t) = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = -\frac{\partial}{\partial p}\dot{p}(q,p,t) \qquad (9.20)$$

which allows to rewrite Eq. (9.19) as follows

$$\frac{\partial f}{\partial t} + \dot{q}\frac{\partial f}{\partial q} + \dot{p}\frac{\partial f}{\partial p} = 0 \qquad (9.21)$$

Distribution function in phase space and kinetic equation

In accelerator physics this equation is often called the *Vlasov* equation. It is a partial differential equation which is not easy to solve in most of the cases.

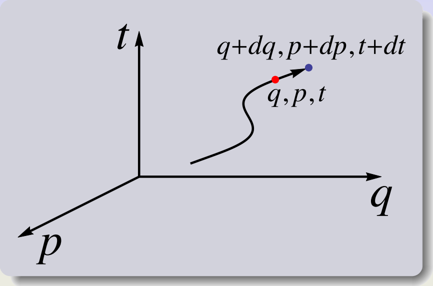
In case of n degrees of freedom, with the canonical variables q_i and p_i, $n = 1, 2, \dots, n$, the distribution function f is defined as a density in $2n$-dimensional phase space and depends on all these variables, $f(q_1, \dots, p_1, \dots, t)$. The Vlasov equation takes the form

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \right) = 0\,. \qquad (9.22)$$

Sometimes it is more convenient to normalize f by N, then the integral of f over the phase space is equal to one.

Integration of the kinetic equation along trajectories

The distribution function is constant within a moving infinitesimal element of phase space "liquid".



Consider a trajectory in the phase space, and calculate the difference of f at two close points on this trajectory. We have

$$\begin{aligned} df &= f(q+dq, p+dp, t+dt) - f(q,p,t) \\ &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp \,. \end{aligned} \qquad (9.23)$$

Remember that the two points are on the same trajectory, hence, $dq = \dot{q} dt$ and $dp = \dot{p} dt$. We find

$$df = \frac{\partial f}{\partial t} dt + \dot{q} \frac{\partial f}{\partial q} dt + \dot{p} \frac{\partial f}{\partial p} dt = 0 \qquad (9.24)$$

On the last step we invoked Eq. (9.21). We proved that the function f is constant along the trajectories.

Integration of the kinetic equation along trajectories

The above statement opens up a way to find solutions of the Vlasov equation if the phase space orbits are known. Let $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$ be solutions of the Hamiltonian equations of motion with initial values q_0 and p_0 at $t = 0$, and $F(q_0, p_0)$ be the initial distribution function at $t = 0$. Then the solution of the Vlasov equation is given by the following equations

$$f(q, p, t) = F(q_0(q, p, t), p_0(q, p, t)) \,, \tag{9.25}$$

where the functions $q_0(q, p, t)$ and $p_0(q, p, t)$ are obtained as inverse functions from equations

$$q = q(q_0, p_0, t) \,, \qquad p = p(q_0, p_0, t) \,. \tag{9.26}$$

A Mathematica notebook demonstrates application of this method to the pendulum motion.

Phase mixing and decoherence

Consider an ensemble of linear oscillators with the frequency ω, whose motion is described by the Hamiltonian

$$H(x,p) = \frac{p^2}{2} + \omega^2 \frac{x^2}{2}\,. \tag{9.27}$$

The distribution function $f(x,p,t)$ for these oscillators satisfy the Vlasov equation

$$\frac{\partial f}{\partial t} - \omega^2 x \frac{\partial f}{\partial p} + p \frac{\partial f}{\partial x} = 0\,. \tag{9.28}$$

We can easily solve this equation. The trajectory of an oscillator with the initial coordinate x_0 and momentum p_0 is

$$\begin{aligned} x &= x_0 \cos\omega t + \frac{p_0}{\omega}\sin\omega t \\ p &= -\omega x_0 \sin\omega t + p_0 \cos\omega t\,. \end{aligned} \tag{9.29}$$

Phase mixing and decoherence

Inverting these equations, we find

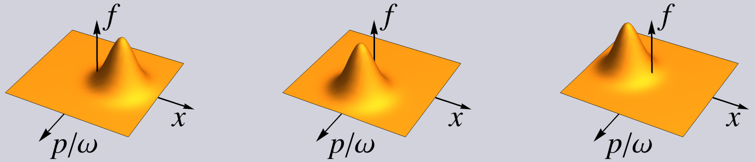
$$
\begin{aligned}
x_0 &= x\cos\omega t - \frac{p}{\omega}\sin\omega t \\
p_0 &= \omega x\sin\omega t + p\cos\omega t\,.
\end{aligned}
\tag{9.30}
$$

If $F(x,p)$ is the initial distribution function at $t=0$, then, according to Eq. (9.25) we have

$$
f(x,p,t) = F\left(x\cos\omega t - \frac{p}{\omega}\sin\omega t, \omega x\sin\omega t + p\cos\omega t\right)\,. \tag{9.31}
$$

Phase mixing and decoherence

This solution describes rotation of the initial distribution function in the phase space. An initially offset distribution function results in *collective* oscillations of the ensemble.



Phase mixing and decoherence

A more interesting situation occurs if there is a frequency spread in the ensemble. Let us assume that each oscillator is characterized by some parameter η (that does not change with time), and ω is a function of η, $\omega(\eta)$.

$$H(x, p, \delta) = \frac{p^2}{2} + \omega(\eta)^2 \frac{x^2}{2} \,. \tag{9.32}$$

We then have to add η to the list of the arguments of f and F, and Eq. (9.31) becomes

$$\begin{aligned} f(x, p, t, \eta) = F\bigg(& x \cos\omega(\eta)t - \frac{p}{\omega(\eta)} \sin\omega(\eta)t, \\ & \omega(\eta) x \sin\omega(\eta)t + p\cos\omega(\eta)t, \eta \bigg) \,. \end{aligned} \tag{9.33}$$

Phase mixing and decoherence

To find the distribution of oscillators over x and p only one has to integrate f over η

$$\hat{f}(x, p, t) = \int_{-\infty}^{\infty} d\eta \, f(x, p, t, \eta) \,. \tag{9.34}$$

The behavior of the integrated function $\hat{f}$ is different from the case of constant ω at large times, even if the spread in frequencies $\Delta\omega$ is small. For $t \gtrsim 1/\Delta\omega$ the oscillators smear out over the phase. This effect is called the *phase mixing* and it results in *decoherence* of collective oscillations of the ensemble of oscillators. We illustrate this effect with a Mathematica notebook.